Alma Mater Studiorum Università di Bologna
Laurea Magistrale in Fisica del Sistema Terra
Corso: Oceanografia Costiera

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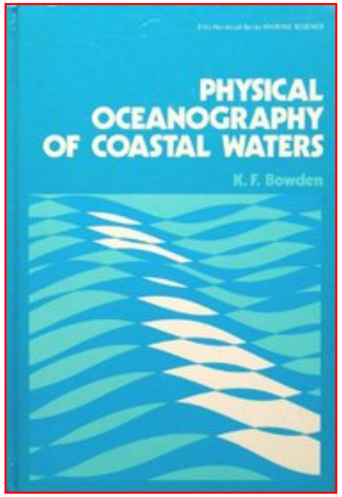
Tides (Part 3)

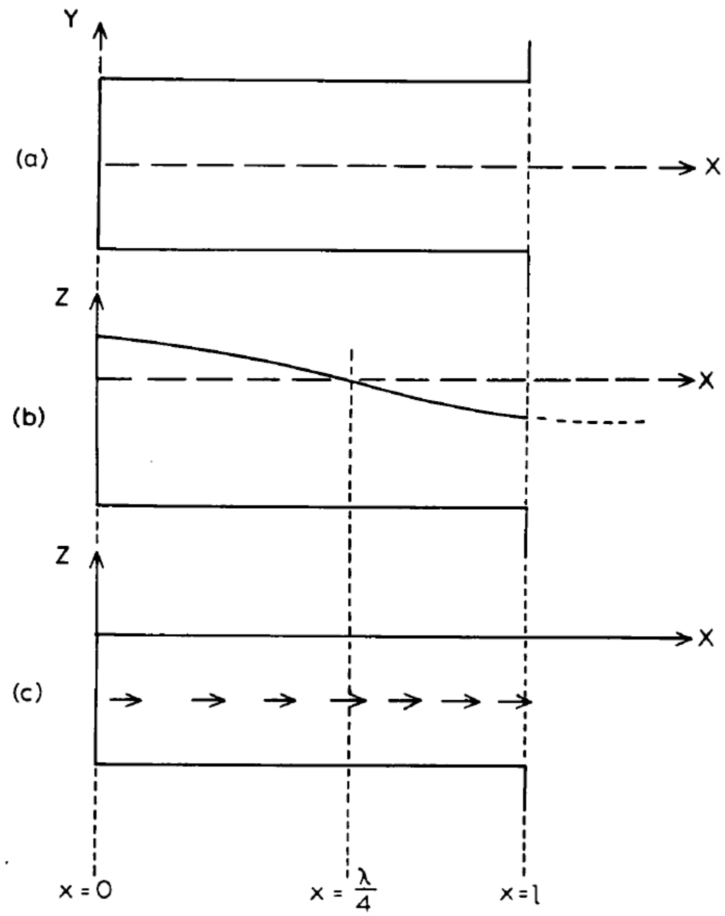
Main references



K.F. Bowden
Physical Oceanography
Of coastal water.
Chapter 2: Tides and tidal currents
Sections 2.5, 2.8

Amphidromic system development

Quantitative assessment



elevation and Currents at t=T/4

Consider a narrow gulf sufficiently small to:
ignore Coriolis force
Ignore frictional effects

x axis along the centre line of the gulf.

$x=0$ at the closed end and $x=l$ at the open end.

At $x=0$ and $u=0$

The equation system is:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

$$H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

Solution is assumed to have the form of a standing wave:

$$u = U \sin \kappa x \cos \sigma t$$

$$\eta = A \cos \kappa x \cos \sigma t$$

$$\boxed{\kappa = 2\pi\lambda} \qquad \boxed{\sigma = 2\pi T} \qquad \boxed{A = \frac{\sigma}{g\kappa}U}$$

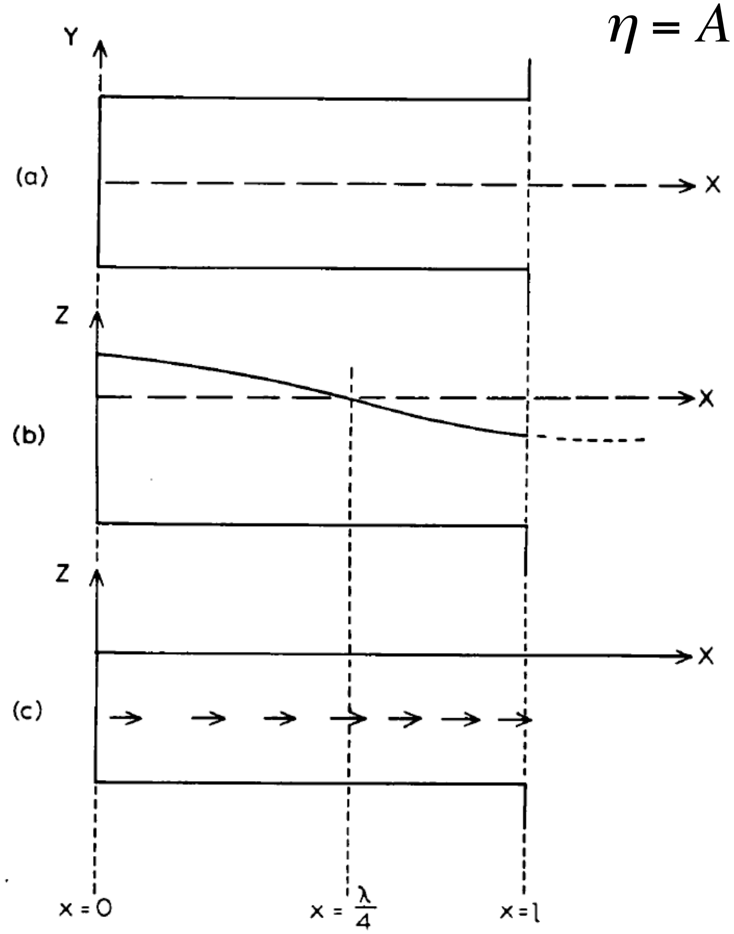
λ=wavelength; T=period

Amphidromic system development

Quantitative assessment

$$u = U \sin \kappa x \cos \sigma t$$

$$\eta = A \cos \kappa x \cos \sigma t$$



We have also (from continuity equation):

$$\sigma^2 = gHU\kappa^2$$

$$\lambda^2 = gHT^2$$

It is assumed that *at* $x=l$ a tidal oscillation of amplitude A_l is prescribed.

$$\eta\big|_{x=l} = A \cos \kappa l \cos \sigma t = A_l \cos \sigma t$$

The amplitude at the head of the gulf is then given by:

$$A = \frac{A_l}{\cos \kappa l}$$

elevation and Currents at t=T/4

Amphidromic system development

Quantitative assessment

$$u = U \sin \kappa x \cos \sigma t$$

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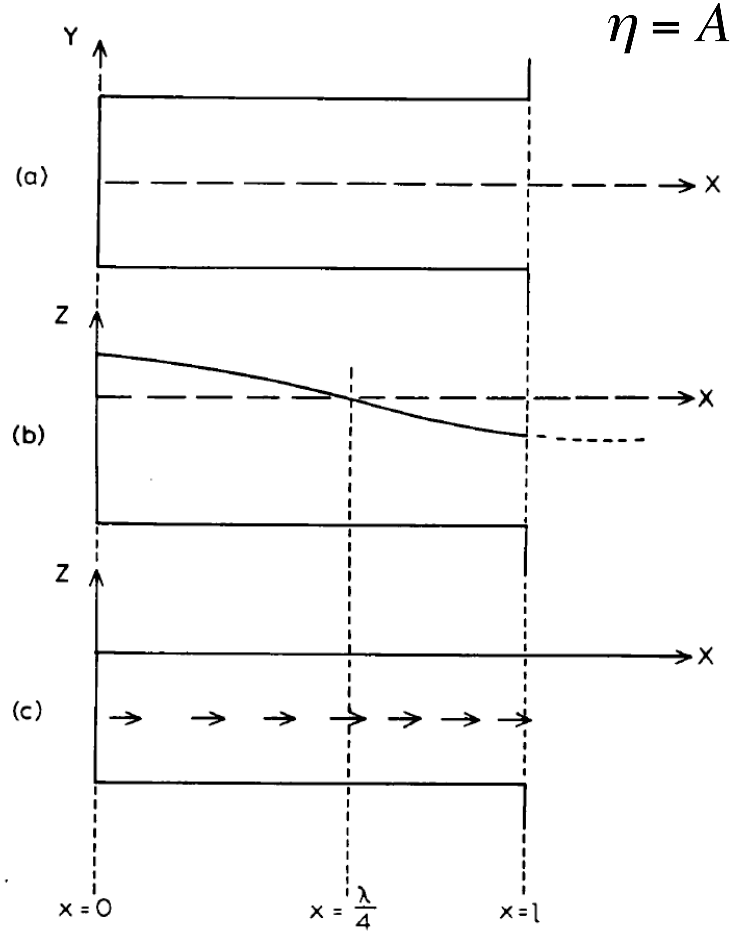


If $\cos \kappa l = 0$ A becomes infinitely large and Resonance occurs at:

$$\kappa l = \frac{\pi}{2}, 3\frac{\pi}{2}, ...(2n-1)\frac{\pi}{2}, n = 1, 2......$$

The 1st resonance at $\kappa l = \frac{\pi}{2}$ corresponds to $l = \frac{\lambda}{4}$

then the basin behave as a quarter wave resonator. This correspond to

$$l = \frac{1}{4} T \sqrt{gh}$$

elevation and Currents at t=T/4

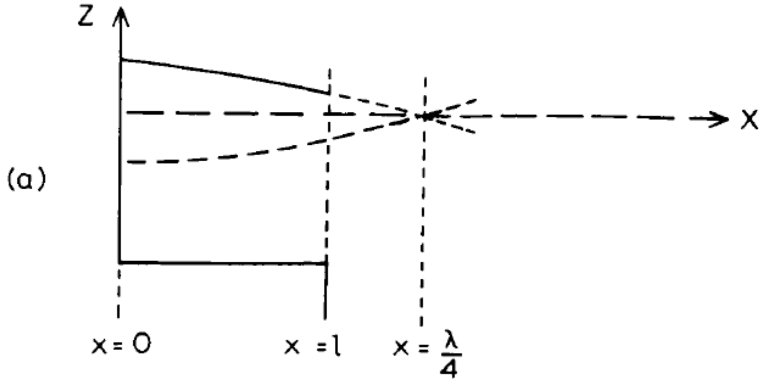
Amphidromic system development

Quantitative assessment

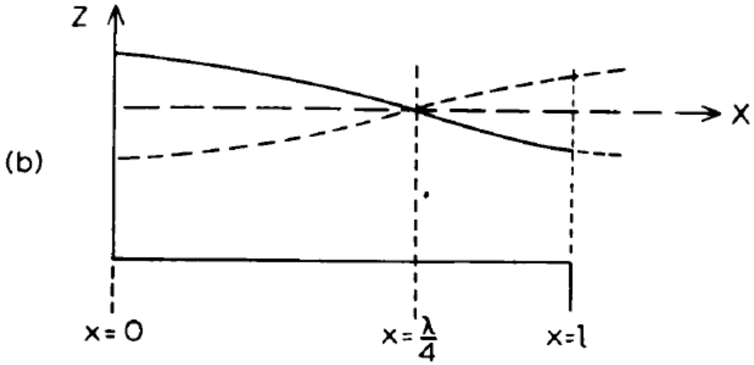
$$u = U \sin \kappa x \cos \sigma t$$

$$\eta = A \cos \kappa x \cos \sigma t$$

$$A = \frac{A_l}{\cos \kappa l}$$



If $l<\lambda/4$:
The phase of the oscillation is the same at all position in the gulf



If $l=\lambda/4$:
There is a nodal line (NOT point!) with tidal range=0 at a distance $x=\lambda/4$ from the head of the gulf seaward.
Where $x>\lambda/4$ the oscillation phase is opposite with repect to points with $x<\lambda/4$

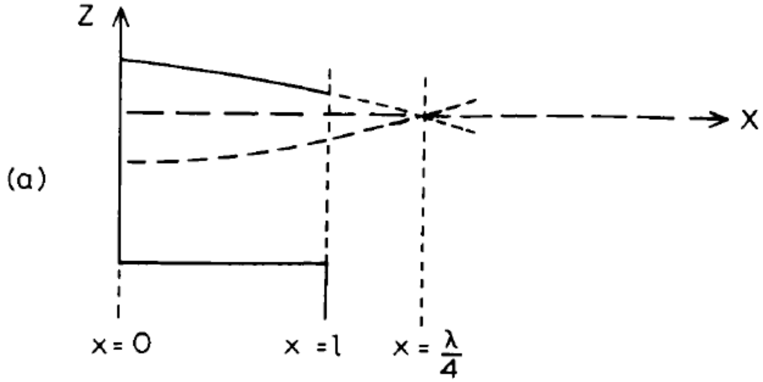
Amphidromic system development

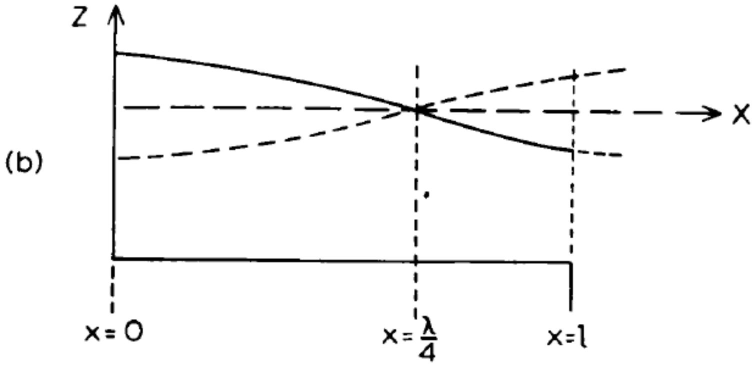
Quantitative assessment

$$u = U \sin \kappa x \cos \sigma t$$

$$\eta = A \cos \kappa x \cos \sigma t$$

$$A = \frac{A_l}{\cos \kappa l}$$





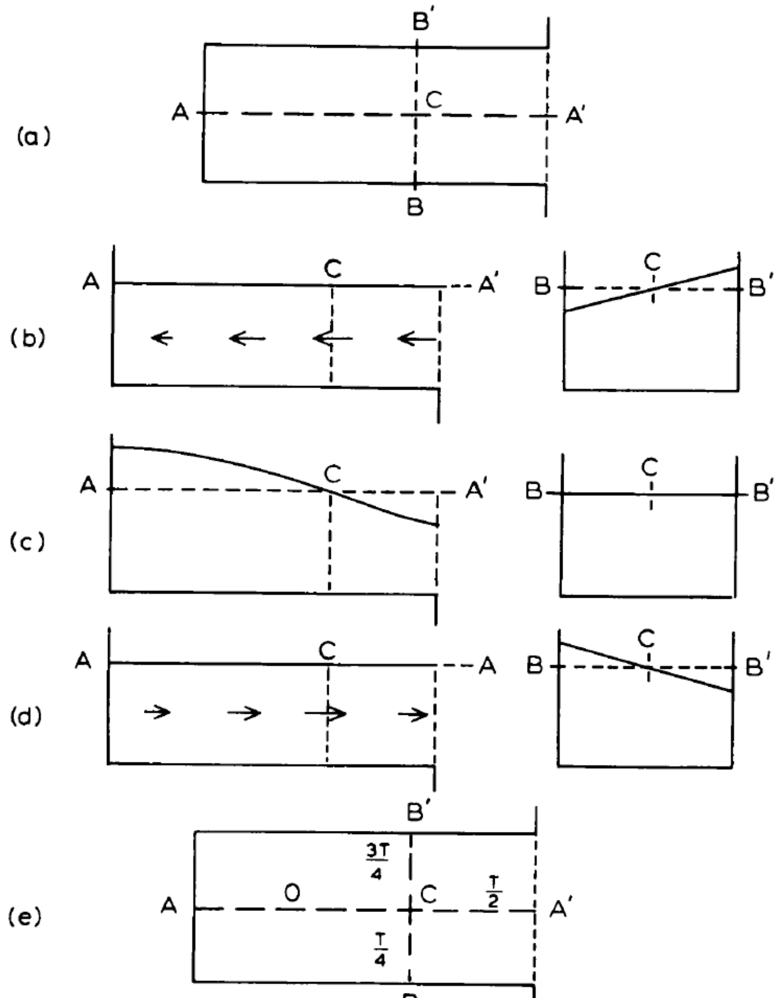
The amplitude at the head of the gulf (in a real tide) never become infinite because as resonance is approached and A increases, the simplifying assumptions made (no friction!!!!) do not hold anymore.

Frictional forces become significant as well as the non linear tems.

Amphidromic system development

Quantitative assessment

<u>Considering Coriolis</u>



The Coriolis force cause a transverse oscillation to be superposed to the longitudinal motion seen before.

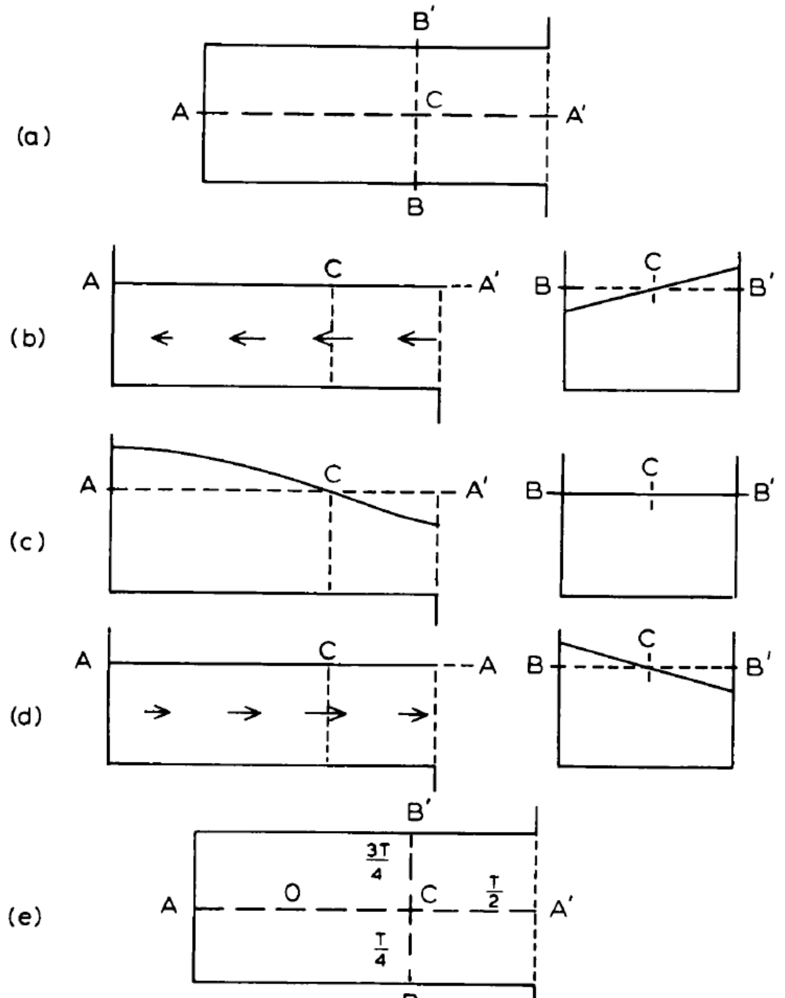
At flood tide: the surface slope upwards to the right of the flow (N. Hemisphere)

Transverse slope vanishes at HT when the current velocity is zero.

At ebb tide slope is opposite to that generated at flood time.

Amphidromic system development

Quantitative assessment
Considering Coriolis



The Transverse slope does not allow anymore for a "*nodal*" line at $x=\lambda/4$ but only for a single "point" (the amphidromic point, C in the figure).

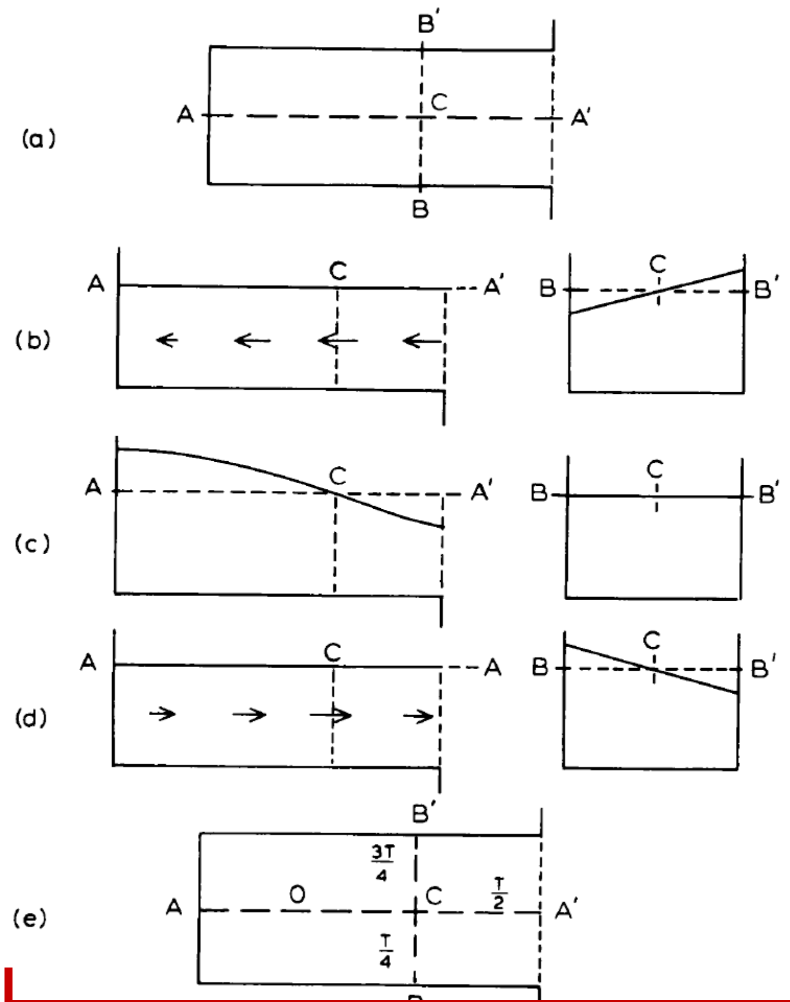
The assumption $v=0$ is not possible in this setup, but it is possible to obtain useful information considering the superposed motion of two Kelvin waves travelling in opposite directions in a channel open at both end, taking as $x=0$ in the point where the elevations due to both waves is always opposite and equal.

The resultant elevation is:

$$\eta = Ae^{y/R}\cos(\kappa x+\sigma t) - Ae^{-y/R}\cos(\kappa x-\sigma t)$$

Amphidromic system development

Quantitative assessment
Considering Coriolis



$$\eta = A\left[e^{y/R}\cos(\kappa x+\sigma t)-e^{-y/R}\cos(\kappa x-\sigma t)\right]$$

At HT we have:

$$\frac{\partial \eta}{\partial t}=0$$

Then:

$$e^{y/R}\sin(\kappa x+\sigma t)+e^{-y/R}\cos(\kappa x-\sigma t)=0$$

This relation define the cotidal lines at Time .t

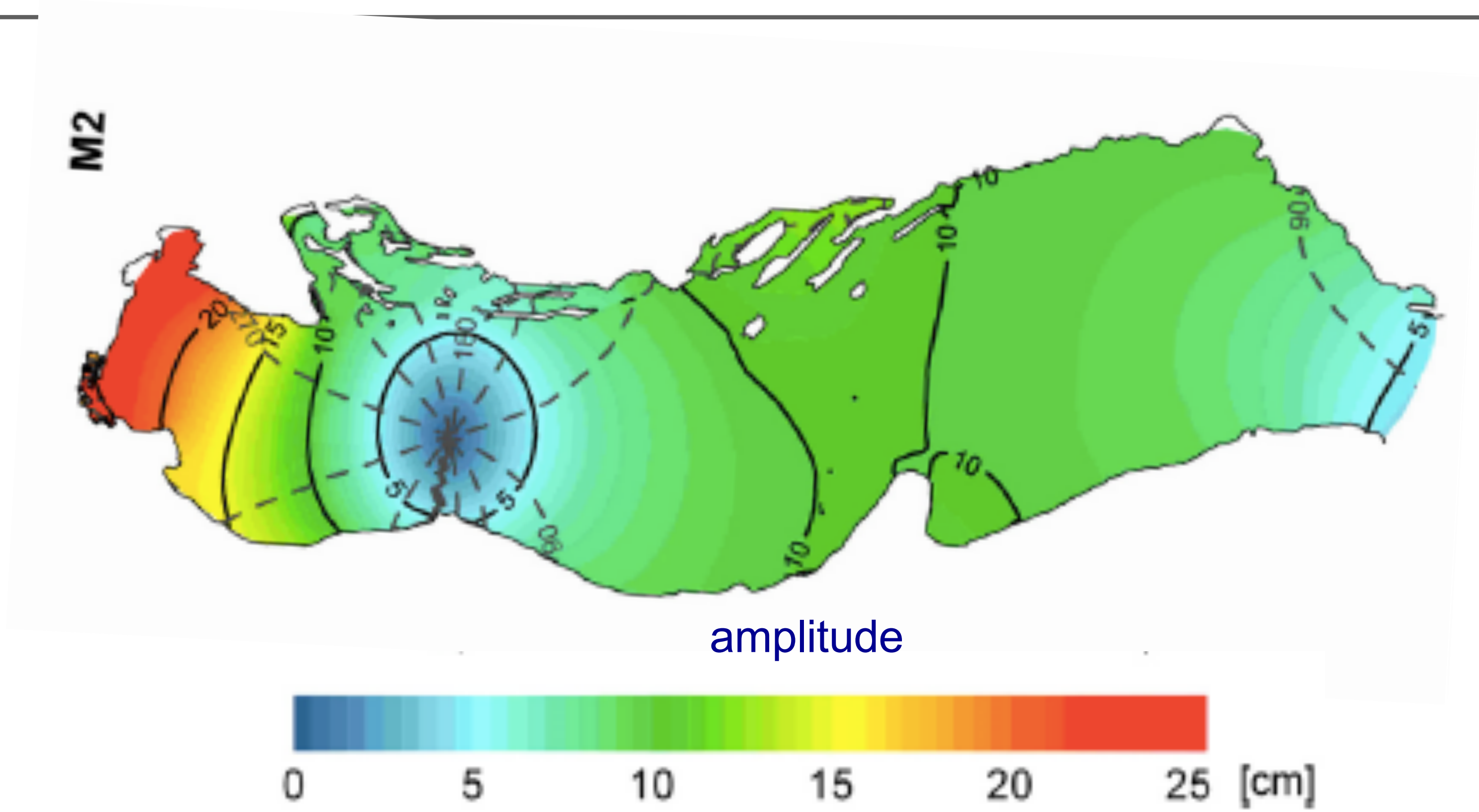
t and can be re-written as:

$$\tan\sigma t=-\frac{\tan\kappa x}{\tanh(y/R)}$$

And then the cotidal line is given by:

$$y=\kappa Rx\cot\sigma t$$

Amphidromic system development

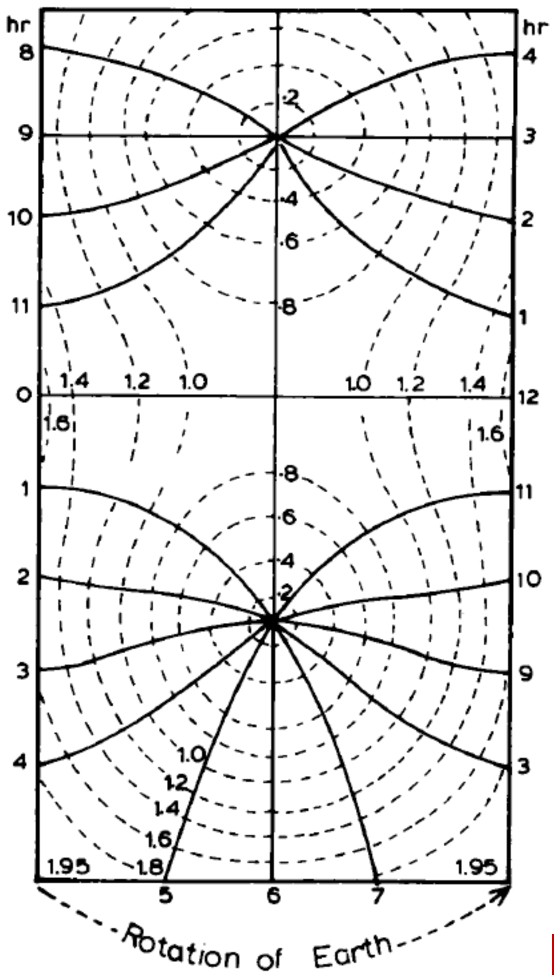


The adriatic sea M2 amphidromic system

Amphidromic system development

A “classic”analytical solution for the standing oscillations produced by a Kelvin wave in a rectangular basin was provided in the 20’s by Taylor

Mouth



(a)

Head

Near the mouth:
Solution correspond to superposition of two Kelvin waves travelling in opposite direction

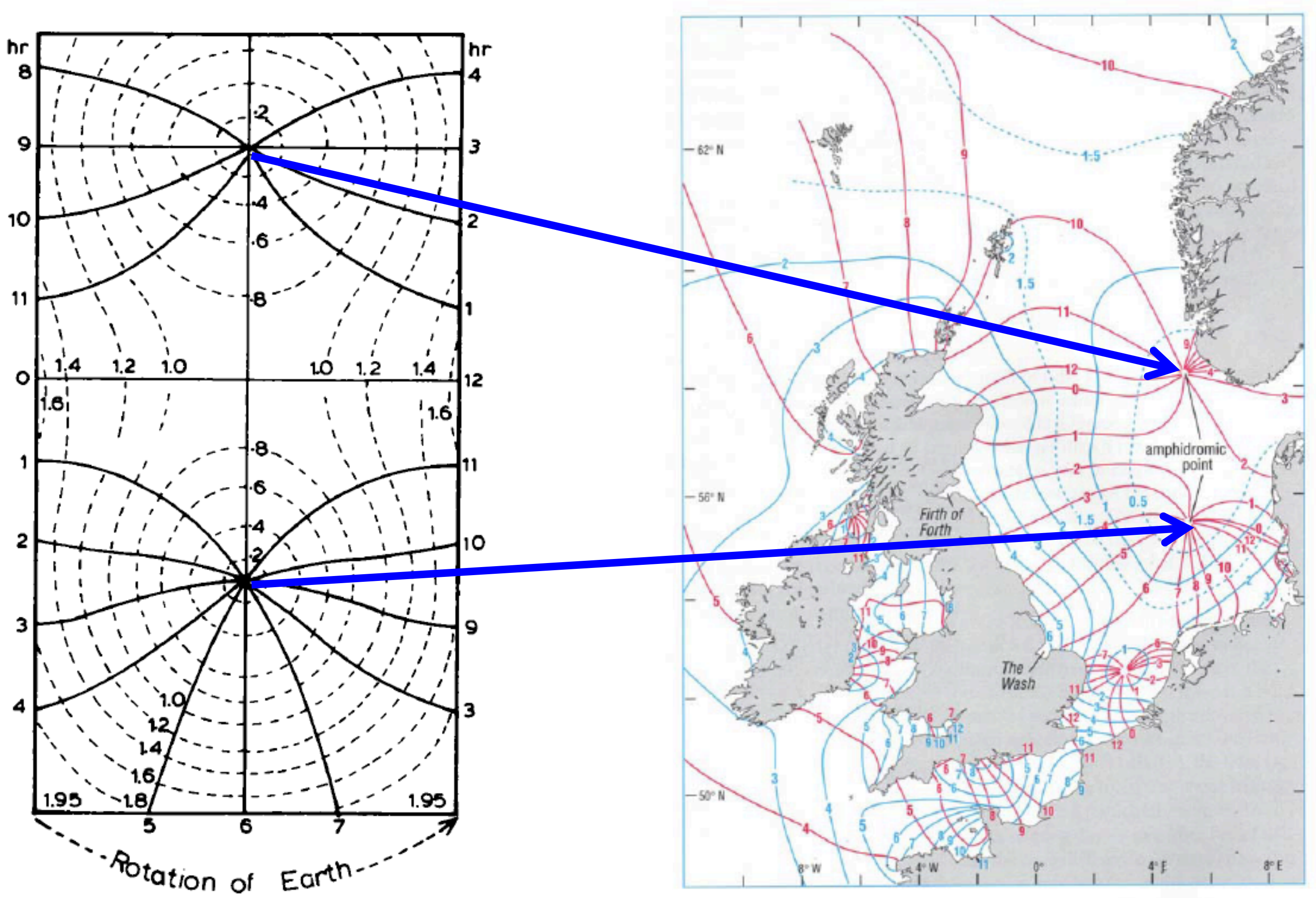
Near the head
Currents normal to the boundary must vanish, then current rotate in direction.

Amphidromic system development

Taylor solution is a 1st approximation of the North Sea tidal system.

Mouth

Head



(a)

Frictional effect on tidal wave

Effects of friction on long waves in the deep ocean is small, but it is important when waves are travelling an shallow depths.

Consider the equations system, used to define the kelvin wave, to which the friction term is added and it is assumed $v=0$

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + C\frac{u}{H}$$

$$fu = -g\frac{\partial \eta}{\partial y}$$

$$H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

Since $v=0$

The friction term in the v equation vanishes and

$\frac{C}{H}u$ becomes $\frac{C}{H}|U|$ where $|U|$ is the amplitude of the depth averaged current.

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + \frac{C}{H}|U|$$

Frictional effect on tidal wave

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} + \frac{C}{H}|U|$$

$$H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

A repetition of the differentiating procedure on the two equations above, carried out before leads to the formulation of the following wave equation.

$$\frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2} - \frac{C}{H}\frac{\partial \eta}{\partial t}$$

A wave equation with a damping term proportional to the particle velocity $\frac{\partial \eta}{\partial t}$

The wave solution has the form: $\eta = Ae^{-\mu x}\cos(\kappa x - \sigma t)$

With $T=\lambda/c$ .Setting $\varkappa=2\pi/\lambda$ and $\sigma=2\pi/\mathrm{T}$ as before.

Frictional effect on tidal wave

$$\eta = Ae^{-\mu x}\cos\left(\kappa x - \sigma t\right)$$

By differentiation and substitution in the wave equation:

$$\sigma^2 = \left(\mu^2 - \kappa^2\right)c$$

$$2\mu\kappa = \frac{C\sigma}{Hc^2}$$

The wave velocity is given by:

$$c_w = c\left(1 - \frac{\mu^2}{\kappa^2}\right)$$ and $$\mu = \frac{1}{2}\frac{Cc_w}{Hc^2} \cong \frac{1}{2}\frac{C}{Hc}$$ since in general: $\mu << \kappa$

The corresponding solution for u is:

$$u = \bar{U}e^{-\mu x}\cos\left(\kappa x - \sigma t - \alpha\right)$$ where

$$\boxed{\bar{U} = \frac{\sigma A}{H\left(\kappa^2 + \mu^2\right)^{1/2}}}$$

$$\boxed{\tan\alpha = \frac{\mu}{\kappa}\bar{U}e^{-\mu x}}$$

Frictional effect on tidal wave

$$\eta = Ae^{-\mu x}\cos(\kappa x - \sigma t)$$

$$u = \bar{U}e^{-\mu x}\cos(\kappa x - \sigma t - \alpha)$$

The equations above describe a damped Kelvin wave travelling in the x direction.

Note that the tidal current is no longer in phase with the elevation but reaches its maximum
A time interval α/σ before the elevation achieve the maximum value.

Frictional effect on tidal wave

Frictional effects and co-oscillation (no Coriolis force)

The standing wave seen before (co-oscillating wave in absence of Coriolis force):

$$u = U \sin \kappa x \cos \sigma t$$

$$\eta = A \cos \kappa x \cos \sigma t$$

May be expressed as the superposition of two progressive wave of same amplitude travelling in opposite direction, one toward the head of the gulf, the other toward the mouth, representing then a wave reflection at the gulf head.

In the resultant oscillation $u=0$ at $x=0$ for all t.

Then

$$\eta = \frac{1}{2} A \left[\cos(\kappa x - \sigma t) + \cos(\kappa x + \sigma t) \right]$$

$$u = \frac{1}{2} U \left[\cos(\kappa x - \sigma t) + \cos(\kappa x + \sigma t) \right]$$

Where (as before)

$$U = \frac{c}{H} A$$

Frictional effect on tidal wave

Frictional effects and co-oscillation (no Coriolis force)

Since we are now including friction the wave of constant amplitude must be substituted by a damped wave (damping factor=μ as before) then the equations for elevation and velocity become"

$$\eta = \frac{1}{2} A \left[e^{-\mu x} \cos(\kappa x - \sigma t) + e^{\mu x} \cos(\kappa x + \sigma t) \right]$$

$$u = \frac{1}{2} U \left[\cos e^{-\mu x} (\kappa x - \sigma t) + e^{\mu x} \cos(\kappa x + \sigma t) \right]$$

HT correspond $\partial \eta / \partial t = 0$ so the timing of high water t_H is given by (from the elevation equation):

$$t_H = \frac{\arctan(-\tan \kappa x \tanh \mu x)}{\sigma}$$

The height oh HT at any position is then given by:

$$\eta_H = A \left[\frac{1}{2} \left[\cos 2\mu x + \cos 2kx \right] \right]^{1/2}$$

Frictional effect on tidal wave

Frictional effects and co-oscillation (no Coriolis force)

$$\eta_H = A\left[\frac{1}{2}\left(\cos 2\mu x + \cos 2kx\right)\right]^{1/2}$$

Note that in the undamped case the amplitude A at $x=0$ would become infinitely large.

$$A = \frac{A_l}{\cos \kappa l}$$

Now, by equating η at $x=l$ to A_l (the prescribed wave amplitude at the mouth of the gulf) it is seen that:

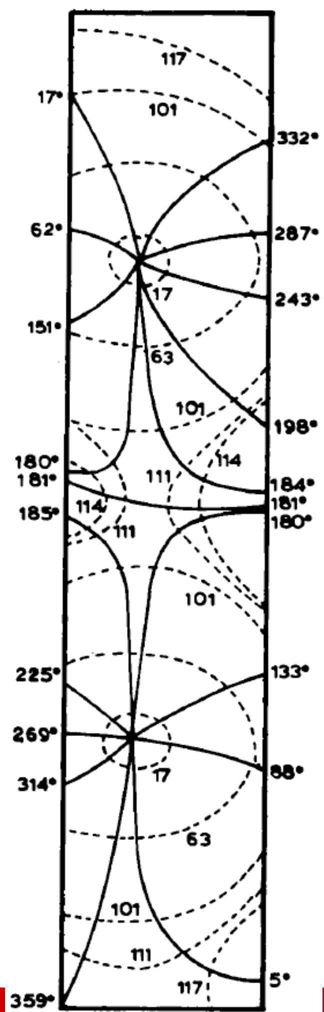
$$A = A_l\left[\cos 2\mu l - 1\right]^{-1/2}$$

A remains finite

Frictional effect on tidal wave

Frictional effects, co-oscillation WITH Coriolis force

Quite difficult analytical treatment. Main characteristics as follows:



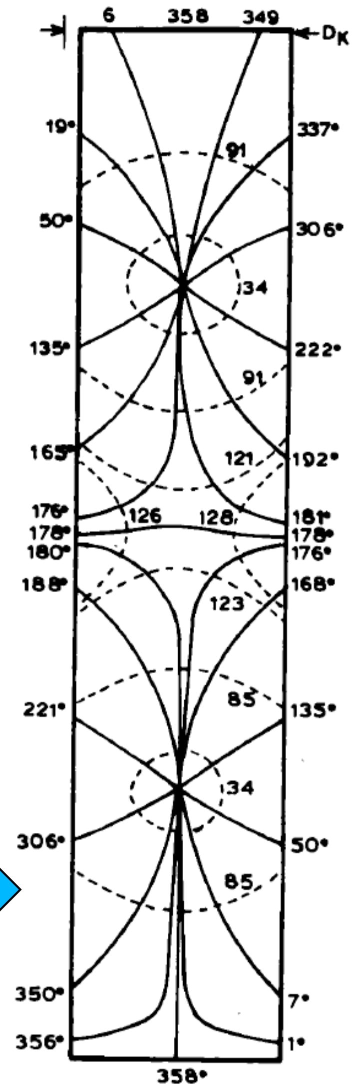
(b) α = 0.056 sec⁻¹

Damped waves: the outgoing wave (left) is smaller that the incoming (right) due to energy dissipation.

The amphidromic point will be then displaced from the basin axis. Difficult analytical treatment. Main characteristics as follows:

Friction

No friction

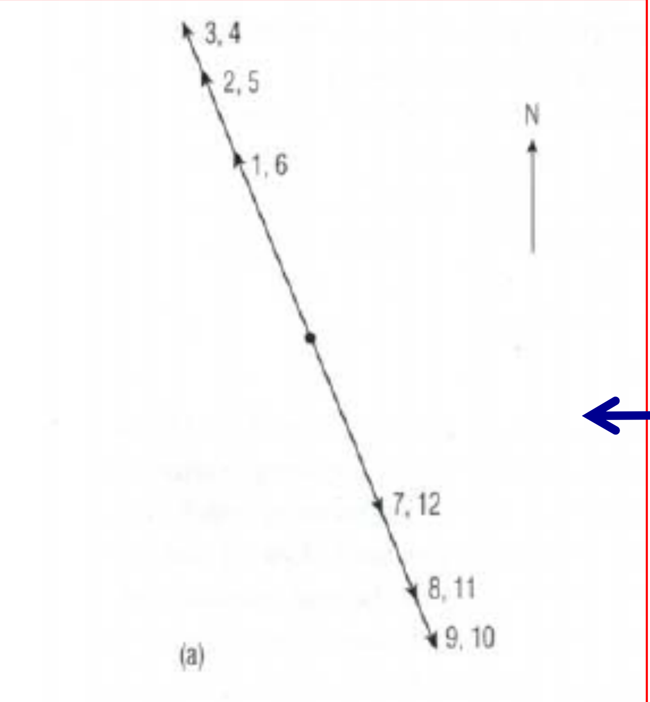


(a) α = 0

Tidal currents

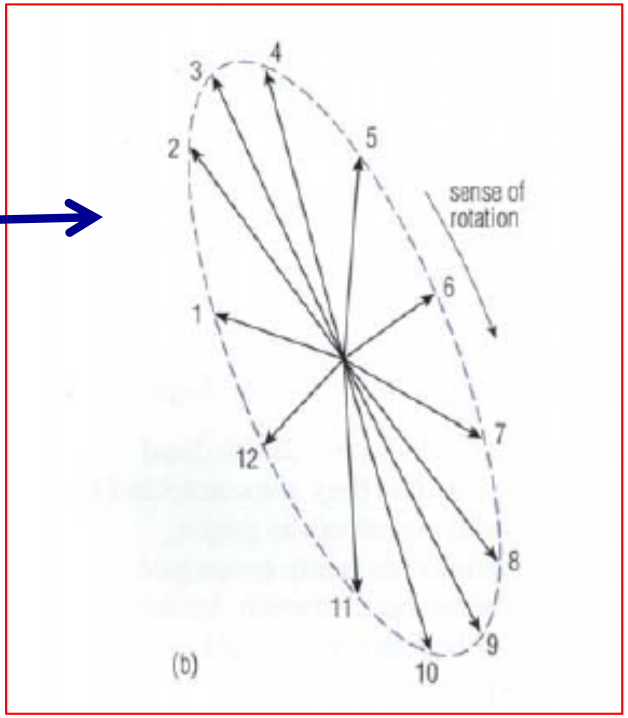
Tidal currents have the same periodicity of the sea level variation and the flow rotate from one direction to the opposite across a tidal cycle (flood/ebb currents)
Such rotation of the current is due to the coriolis force.

Water particles are then following a (more or less) elliptical pattern.



Purely ebb/food System (channel)

Typical ellipse



Tidal currents

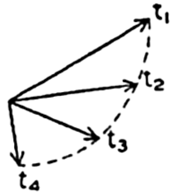
The velocity component (u and v) associated to a given tidal constituent of angular frequency σ can be written as:

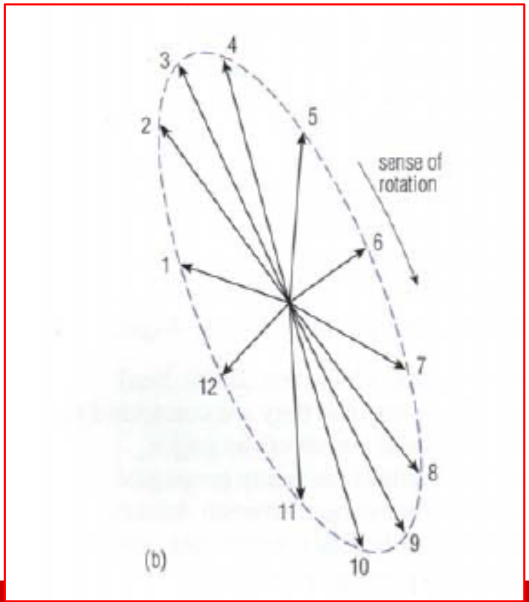
$$u = A_1 \cos\sigma t + B_1 \sin\sigma t$$

$$v = A_2 \cos\sigma t + B_2 \sin\sigma t$$

Where $A_{(1,2)}$,$B_{(1,2)}$ are specific constants.. Plotting the resultant velocity on a vector diagram:

the head vector will trace an ellipse.





it is then possible to transform to axes parallel to the axes (major and minor) of the ellipse

Tidal currents

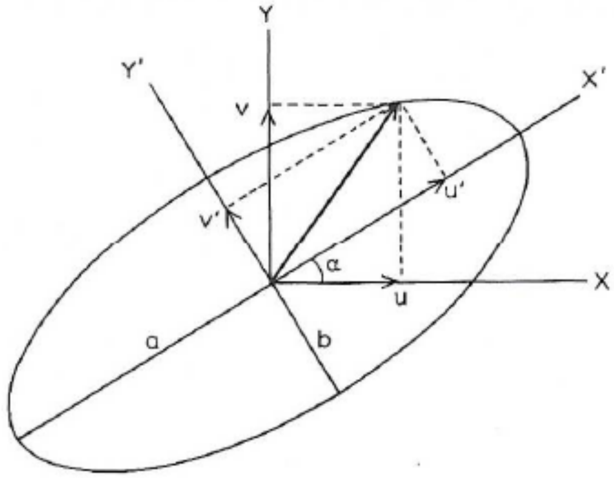
it is then possible to transform to axes parallel to the axes (major and minor) of the ellipse. Denoting the velocity components relative to these axes as u' and v', we can write:

$$u' = a\cos(\sigma t + \varepsilon)$$

$$v' = b\cos(\sigma t - \varepsilon)$$

a,b=semi major and semi minor axes of the ellipse

ε=phase angle

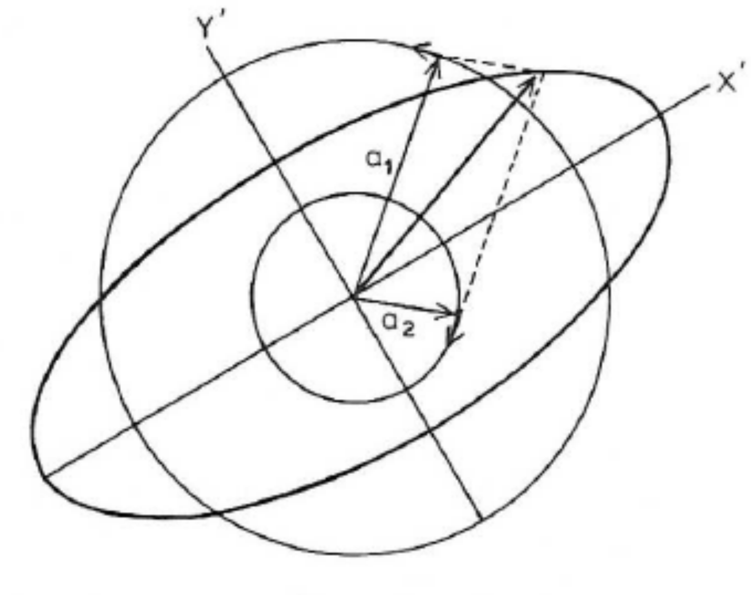


with:

$$\frac{u'^2}{a^2} + \frac{v'^2}{b^2} = 1$$

Tidal currents

An alternative method of representing a rotating current is as superposition of two circular components rotating in opposite directions:



Considering the motion given by:

$$u_1 = a_1 \cos \sigma t \qquad v_1 = a_1 \sin \sigma t$$

The resultant motion is represented by a vector of constant length a_1 rotating to the left with increasing t

Similarly:

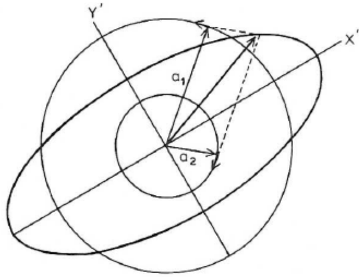
$$u_2 = a_2 \cos \sigma t \qquad v_2 = -a_2 \sin \sigma t$$

Is represented by the constant length vector a_2 rotating to the right'

The motion resulting by the composition of vectors a1 and a2 is therefore described by:

$$u = u_1 + u_2 = (a_1 + a_2) \cos \sigma t \qquad v = v_1 + v_2 = (a_1 - a_2) \sin \sigma t$$

Tidal currents



$$u = u_1 + u_2 = (a_1 + a_2)\cos\sigma t \qquad v = v_1 + v_2 = (a_1 - a_2)\sin\sigma t$$

The semi-major (a) and semi-minor (b) axes are given by:

$$a = a_1 + a_2 \qquad b = a_1 - a_2$$

Rotation to the left if:

$$a_1 > a_2$$

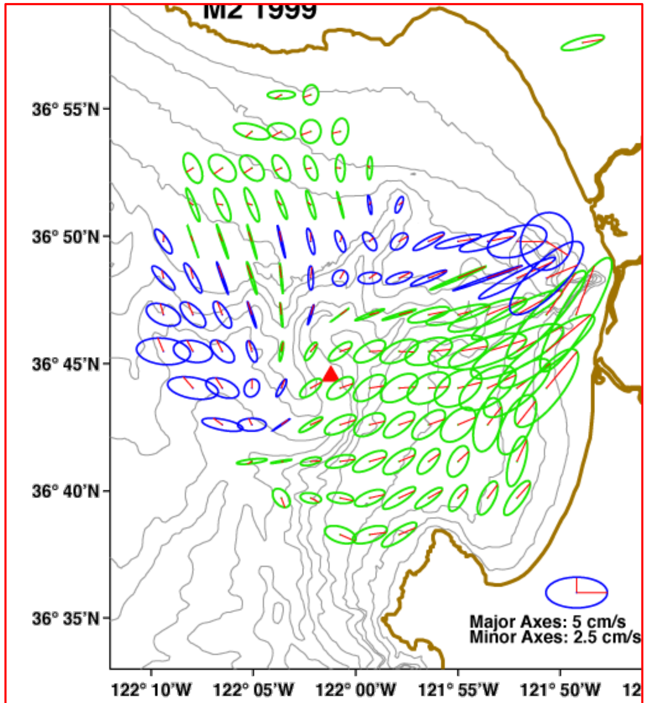
Rotation to the right If

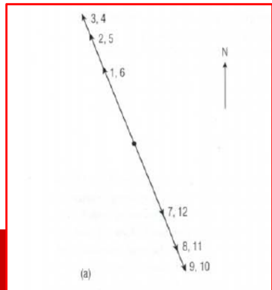
$$a_1 < a_2$$

If

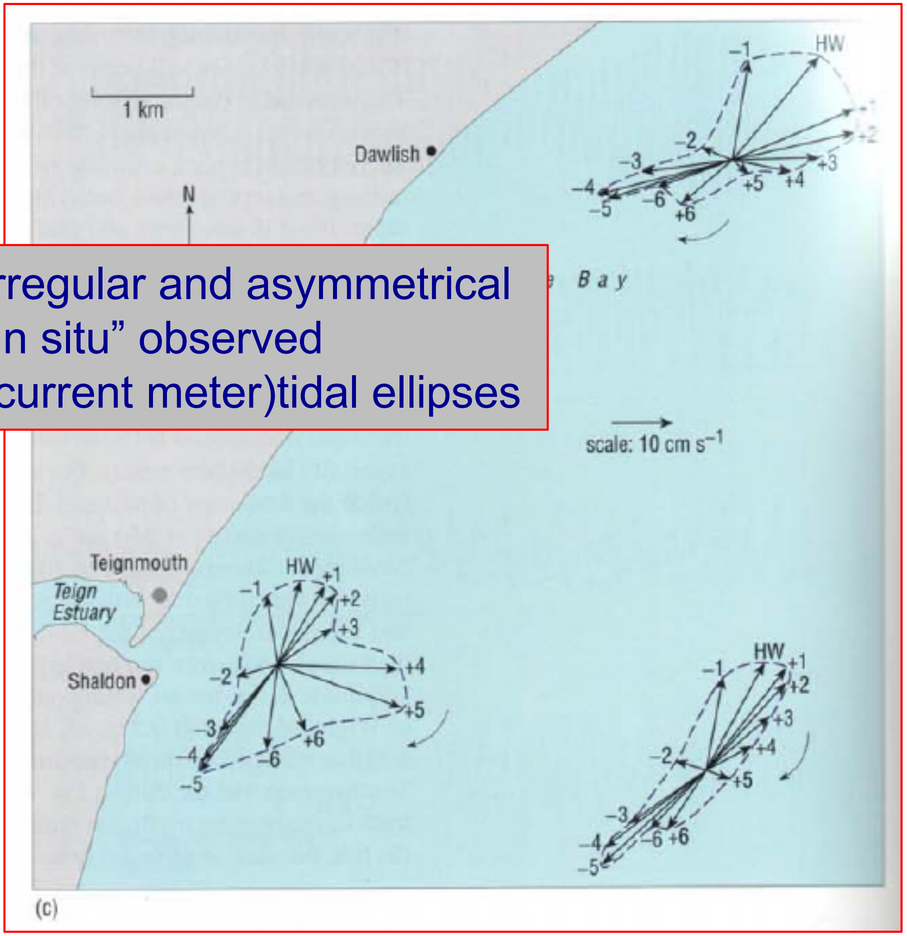
$$a_1 = a_2$$

Rectilinear motion in the x direction (purely ebb-flood motion)





Tidal currents



Irregular and asymmetrical "in situ" observed (current meter)tidal ellipses

Tidal currents rotation pattern is clearly modified By coastline geometry, bottom topography and local weather conditions.

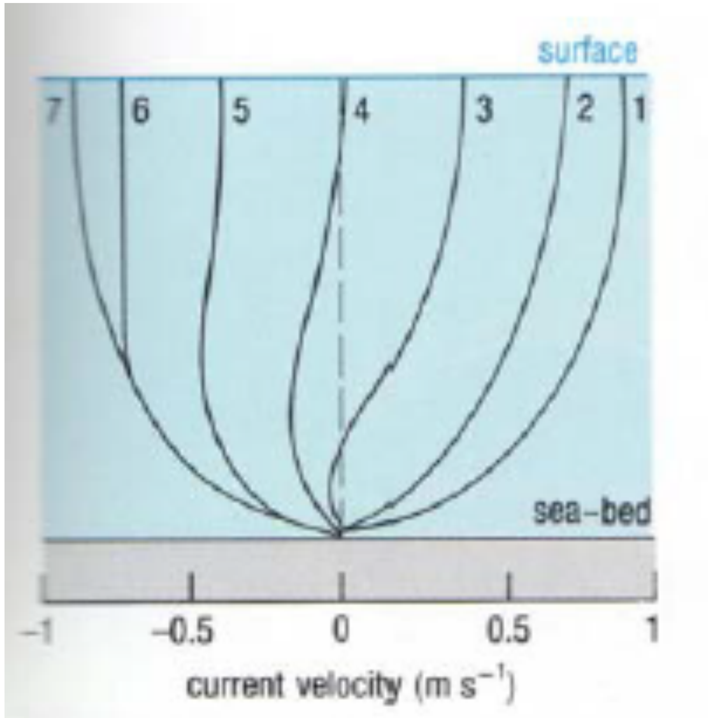
As a result of the distortion of the elliptical pattern, Is the generation of a residual current: A long term net movement of water in a well defined direction

Tidal currents

Tidal currents vertical profile

bottom stress due to friction determines retardation of the flow toward the bottom and involve also a vertical variation of the phase (current reverses earlier near the bottom then at surface.

The dynamics is obviously regulated by the bottom boundary layer development (see specific lecture)



Vertical velocity profiles across
A tidal cycle

Tidal energy and dissipation

The rate of trasmission of energy of a tidal wave across a vertical section is given by

$$p'u$$

Where p' is the excess pressure and u is the velocity normal to the section.

for a long wave $p'=\varrho g\eta$, Then: $p'=\varrho g\eta u$

Integrating vertically from $z=-H$ to $z=\eta$ and assuming $\eta<<H$ ithe rate of energy transmission across a section of unit width is:

$$E = g\rho H\eta\bar{u}$$

For a given harmonic constituent let:

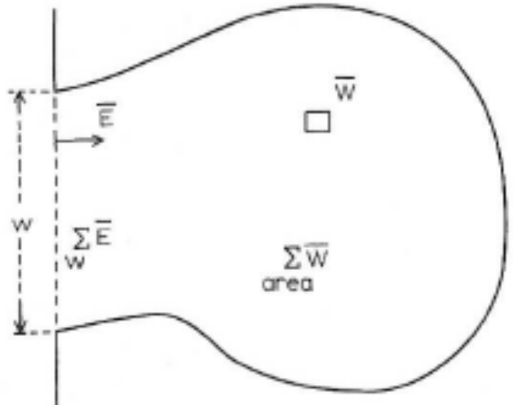
$\eta = A\cos\sigma t \quad \bar{u} = U\cos(\sigma t-\delta)$ δ:phase difference between elevation and current

Then the mean rate of transmission of energy over a tidal period is:

$\bar{E} = \frac{1}{2}g\rho HAU\cos\delta$ Integrating across the width of the section $\int_W \bar{E}$ represent the total flux of energy trough the section (to be totally dissipated if the section delimits a enclosed basin).

Tidal energy and dissipation

$$\bar{E} = \frac{1}{2} g\rho HAU \cos\delta$$



An alternative calculation of the rate of dissipation is based on the work done by bottom friction. The rate of trasmission of energy of a tidal wave across a vertical section is given by

$W = \tau_B U_b = C_d \rho |U_b| U_b^2$ with $\tau_B = C_d \rho |U_b| U_b$ and U_b=bottom current amplitude

Assuming complete dissipation of energy, the dissipation should balance the energy flow into the basin:

$$\int_A W = \int_W \bar{E}$$

Where $\int_A$ denotes the integral over the basin area and $\int_W$ the integral across the section width.

Tidal energy and dissipation

World wide tidal (M2) energy dissipation: $1.7\ 10^6\ MW$ (total energy in the world tides dissipated in about two days)
Over continental shelf:

Table 2.2 – Flux of M_2 tidal energy into coastal seas.

Region	10^4 MW	Percentage of total
N.W. European Shelf Seas	19	12
Hudson Strait	12	8
Patagonian Shelf of S. America	13	9
Other Atlantic shelf regions	9	6
Bering Sea	3	2
Sea of Okhotsk	21	13
Other Pacific shelf regions	29	19
Northern Australian Shelf	18	12
Indian Ocean shelf regions	23	15
Arctic and Antarctic regions	7	4
Total	154	100